

Controlling near shore nonlinear surging waves through bottom boundary conditions

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Instead of taking the usual passive view for warning of near shore surging waves including extreme waves like tsunamis, we aim to study the possibility of intervening and controlling nonlinear surface waves through the feedback boundary effect at the bottom. It has been shown through analytic result that the controlled leakage at the bottom may regulate the surface solitary wave amplitude opposing the hazardous variable depth effect. The theoretical results are applied to a real coastal bathymetry in India.

I. INTRODUCTION:

Near shore coastal regions often witness surging of the approaching waves including extreme events like tsunamis [1]. Such a natural phenomenon has also been observed, though in a miniature scale in few rivers around the world as bore waves [2–4]. Famous examples are the river Seine in France and the river Hoogli in India [1, 2]. Such surging waves are suspected to be caused by nonlinear gravity waves, propagating over a decreasing depth bathymetry towards the shore or along upstream river. Such events which can often trigger extremely hazardous effects have attracted intense attention over centuries and have been studied extensively from both theoretical and practical points of view [1]. The main emphasis of the investigations was to work towards the development of early warning systems for minimizing the loss of human lives. The present development of the tsunami warning system has definitely been reached to a satisfactory level[5]-[10].

However, there are few situations where the installation of a passive warning system is not enough, while the demand is for more active intervention. This is particularly true for example, in protecting nuclear reactors and related installations, which are located usually at the vicinity of the sea shore due to logistic reasons, against the tsunami threat. As we know the tsunami of 2004 which played devastating effects spreading over many countries was a potential threat to the nuclear reactor at Kalpakkam in India. The tsunami of 2010 inflicted real calamities in Fukushima nuclear reactors in Japan [11, 12].

In a relatively smaller scale, the near shore waves and bore waves caused many devastating effects to the coastal habitats and in-land rivers throughout the centuries. Therefore along with the traditional warning systems, it is desirable to find ways and means geared towards possible invasive procedures for taming of such hazardous wave phenomena. There are few suggestions for effective interventions, like plantation of Mangrove trees along the coastal lines [13], installation of breakwaters at strategic positions [14]-[17], stoppage of erosion by concrete boulders etc.

However, these are mostly indirect ways to counter the surging waves, while we lack proposals on directly attacking the problem, perhaps with the exception of the proposed bubble method, aiming to stop the incoming waves by a stream of fast and strong counter-waves, mixed with air bubbles [18]. Though the last method was proposed more than fifty years back, its feasibility and effectiveness has not been established yet. The attenuation of incident water waves by a curved vane like structure positioned beneath or at the surface of a body of water is described in a Patent [19] where the detailed design of the structure is given. An attempt was made to reduce the devastating effects of a tsunami waves by single and double submerged barrier was done in Tel Aviv University[20]. They performed their experiments in a basin 5 m in length and 10.5 cm in depth. The wavelength of the generated wave was about 3 m, which allows referring to it as a tsunami.

Our aim here is to put forward an innovative proposal based on a theoretical study on the effect of a feedback boundary control at the bottom on the surging surface wave amplitude. The governing nonlinear equations describing unidirectional gravitational waves are derived from the basic hydrodynamic equations at the shallow water regime. The key factor responsible for surging of the nonlinear waves approaching to the shore (or in upstream rivers) is the decreasing depth bathymetry, which triggers the amplitude surge of the surface waves inversely proportional to the water depth which diminishes continuously along the wave propagation towards the shore.

Our strategy is to study first the effect of the bottom boundary condition on the nonlinear solitary surface waves of the well-known perturbed KdV equation, propagating in shallow water of constant depth. The vertical fluid velocity

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at the bottom is taken as a function of surface wave profile, to identify subsequently through theoretical analysis, the optimal case inducing maximum amplitude damping to the surface waves. This knowledge is applied through slowly varying bathymetry, which without the leakage condition, as we know would result to solitary wave solution with increasing amplitude with the water depth decreasing along its propagation. However when the controlled bottom leakage with optimal feedback wave profile is imposed, the surging amplitude of the wave meets the counter damping effect, resulting to a managed propagating waves towards the shore with reduced hazardous effect due to the effective damping of the wave.

We would like to emphasize that there could be various natural bottom boundary effects inducing damping of the surface wave amplitudes, like porosity[21]-[30], irregularities, uneven heights, periodic topography, friction[31, 32] apart from the fluid viscosity [33] etc. while the long obstacle can induce fission of the solitary waves [31]. However our aim here is to induce damping effect artificially through controlled mechanism.

The privilege of our theoretical result is the exact nature of the solutions we obtain, in spite of the variable depth bathymetry, which is rather a rare achievement. Our theoretical results with exact solutions allow to extract finer details and precise predictions. Our findings are extended to cover different cases of the controlled bottom leakage conditions, ranging from space dependent to time dependent, from vanishing of effective leakage velocity to a desirable leakage conditions etc. Our theoretical findings for the possible control of the surging waves like tsunamis and bore waves based on our exact results are applied next to real sea shore bathymetries. We have focused in particular on two high risk coastal zones of bay of Bengal near the city in Chennai of south India as presented in a recent in depth study of the subject [34]. Our analysis shows that a significant upsurge could have been experienced by a future surging wave approaching towards these coastal points. For example at the identified northern coastal point (N $13^{\circ} 10.5'$ - E $80^{\circ} 18.75'$) a wave of nearly 1 meters built at a distance of 10.5 km from the shore would have been developed to a killing height of 30 meter at the shore without any control. Similarly at a southern point (N $13^{\circ} 0'$ - E $80^{\circ} 16.2'$) the bathymetry would induce equally devastating upsurge for a wave of 1 m created at 11.1 km away, to develop into a 30 meter killer wave at the shore. Applications of feedback controlled method through bottom boundary condition, that we propose here is found to be able to regulate such upsurging waves to a considerable extent, minimizing its hazardous effects. In particular, the surging waves at the northern point could be regulated at the height of 1.23 meters if the leakage installation could be made starting from a distance of 900 meters from the shore. A smaller distance could result to a higher amplitude though significantly lesser than that without control. Similarly the surging waves at the southern coast could be controlled to a wave amplitude of 0.4 meter, if the installation starts at a distance of 900 meters. Thus the hazardous effects of tsunami like surging waves could possibly be neutralized to some extent through a controlled bottom leakage condition tuned by a linearly dependent wave profile which we found to be optimal, created through a feedback mechanism.

The paper is organized as follows. In section II, the shallow water problem in constant depth with a specific class of feedback bottom BC with controlled leakage is considered. Corresponding surface wave evolution equation is derived for different leakage conditions and solved for its solitary wave solutions, through Bogoliubov-Mitropolsky approximation. In the next section variable depth problem is taken up, for the optimally controlled leakage and the related wave equation is derived extracting exact solution for a tuned balance between the effects of leakage and the variable bathymetry.

Our theoretical findings are applied to a real near shore bathymetry data and our predictions are checked for different installation distance for the controlled mechanism which is contained in sec IV. The extension of this exact result is considered in the appendix (sect. VI) for a particular bathymetry and leakage velocity obtaining again a solvable equation with exact solution. This section also includes time dependent leakage and leakage that contains both wave profile dependent and independent part. Section V is the concluding section followed by the bibliography.

II. EFFECT OF FEEDBACK BOTTOM BOUNDARY CONDITION ON NONLINEAR SURFACE WAVE IN CONSTANT DEPTH

The purpose of this section is to study the effect of bottom BC with controlled leakage, designed with a feedback from the surface wave, where the leakage function would depend on the wave profile and its spatial derivatives. As is well known that, the nonlinear free surface gravity waves propagating in a shallow water in constant depth with the traditional hard bed boundary condition in the form of solitary waves retain their constant amplitude profile with a high degree of stability. However, when the boundary condition is changed to a leakage function dependent on the wave profile itself, as we find here, the solitary waves propagating on the surface suffer an amplitude damping along its propagation. Different forms of the leakage velocity function at the bottom induce different types of damping. Such a controlled leakage at the bottom may be arranged using a functional feedback from the profile of the wave appearing on the surface over that location and at that instant of time. Our motivation for this study is to analyze different damping effects corresponding to different leakage functions and identify the case when the damping would

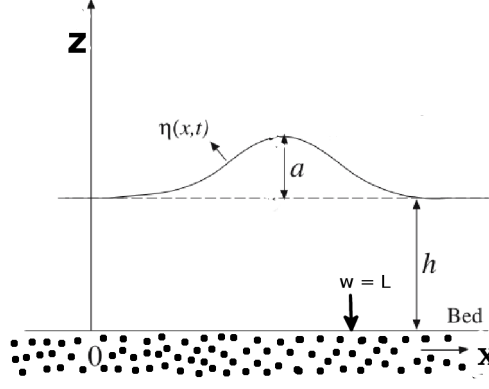


FIG. 1: Solitary wave in shallow water of constant depth with leakage at the bottom

be maximum, which is the most desirable feature in the present context.

In the following subsections we derive the corresponding free surface wave equation and investigate the nature of the solitary wave solution with damping caused by different cases of the bottom leakage condition.

A. Surface wave evolution equation with leakage boundary condition

We consider here the shallow water nonlinear surface-gravity wave, propagating along the positive x -direction in a constant water depth with the viscosity and the surface tension of the fluid, which is assumed to be incompressible, are neglected in what follows. We start from the dimensionless basic hydrodynamic equations [35]:

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \delta^2[w_t + \epsilon(uw_x + ww_z)] = -p_z, \quad (1)$$

along the x and the z axis, respectively, which are reducible from the Euler equation in the present case.

Here u, w, p, η are horizontal and vertical fluid velocity components, pressure and the surface wave profile, respectively, with the subscripts denoting partial derivatives. ϵ is the amplitude parameter defined by $\epsilon = \frac{a}{h}$ and $\delta = \frac{h}{l}$ is the shallowness parameter, expressed through the maximum amplitude a , the water depth h and the wavelength l (see FIG. 1). ϵ and δ are natural parameters supposed to be small, which is consistent with the long wave and the shallow water limit. The continuity equation of the fluid yields

$$u_x + w_z = 0. \quad (2)$$

Nonlinear variable boundary conditions, valid at the free boundary $z = 1 + \epsilon\eta$, on the other hand, gives

$$p = \eta, \quad w = \eta_t + \epsilon u \eta_x, \quad (3)$$

while we take the boundary condition for the vertical component of the water velocity at the bottom: $z = 0$ as

$$w = -\epsilon \tilde{\alpha} G(\eta, \eta_x, \dots), \quad (4)$$

where $G(\eta, \eta_x, \dots)$ is assumed, in general, to be an arbitrary function of η and its spatial derivatives and α' is a positive constant with ϵ being a small parameter as defined above. It is important to note here, that usual hard bed scenario with no leakage one would have $w = 0$ at the bottom whereas in our choice the nontrivial leakage function G may depend functionally on the surface wave profile which could be designed through a feedback route, sensing the surface movement. The leakage is considered here to be in the ϵ order. Note that the negative sign in equation (4) appears because the leakage velocity occurs along the negative z -direction, i.e, vertically downward. In order to model shallow water solitary waves, there must be an appropriate balance between nonlinearity and dispersion, i.e, $\delta^2 = O(\epsilon)$ as ϵ tends to zero. Thus for any δ , there exists a region in (x, t) - plane with ϵ tending to zero, where this balance remains valid. This region of our interest may be defined by a scaling of independent variables as $x \rightarrow \frac{\delta}{\sqrt{\epsilon}}x$, $t \rightarrow \frac{\delta}{\sqrt{\epsilon}}t$ and $w \rightarrow \frac{\sqrt{\epsilon}}{\delta}w$ for any values of ϵ and δ . The set of equations (1-4) thus becomes,

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \epsilon[w_t + \epsilon(uw_x + ww_z)] = -p_z, \quad u_x + w_z = 0, \quad (5)$$

together with the boundary conditions

$$p = \eta, \quad w = \eta_t + \epsilon u \eta_x, \quad (6)$$

$$w = -\epsilon \alpha G(\eta, \eta_x, \dots). \quad (7)$$

valid at the free surface and at the bottom, respectively, where $\alpha = \tilde{\alpha} \frac{\delta}{\sqrt{\epsilon}}$, with a net outcome of the transformation is to replace δ^2 by ϵ in equations (1-4). Introducing a new frame of reference with stretched time $\xi = x - t$, $\tau = \epsilon t$, we seek an asymptotic solution of the system of equations and boundary conditions in the form

$$q(\xi, \tau, z; \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n q_n(\xi, \tau, z), \quad \eta(\xi, \tau; \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \eta_n(\xi, \tau), \quad (8)$$

where q (and related q_n) represents each of the functions u, w and p for the corresponding expansion.

Now to deduce the final evolution equation from the set of complicated nonlinear equations (5-7) involving several variables, we have to make the asymptotic multi-scale expansions as explained above. Below, we carry out an explicit order by order calculation to demonstrate the process.

1. Result at ϵ^0 order

At ϵ^0 order the above set of equations (5)-(7) is reduced respectively to the following set

$$u_{0\xi} = p_{0\xi}, \quad p_{0z} = 0, \quad u_{0\xi} + w_{0z} = 0 \quad (9)$$

$$p_0 = \eta_0, \quad w_0 = -\eta_{0\xi}, \quad (10)$$

$$w_0 = 0, \quad (11)$$

with equation (10) valid at $z = 1$ and (11) at $z = 0$. These equations lead to the solutions expressed through η_0 as $p_0 = \eta_0$, $u_0 = \eta_0$, $w_0 = -z\eta_{0\xi}$, with the appearance of η caused only by the passage of the wave has been imposed, i.e., $u_0 = 0$, whenever $\eta_0 = 0$.

2. Result at ϵ order

In this order of approximation, two free boundary conditions at $z = 1 + \epsilon\eta$ are evaluated by performing Taylor expansions of the functions u, w, p around the point $z = 1$. Consequently the following set of equations are obtained from (5)-(7):

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}, \quad p_{1z} = w_{0\xi}, \quad u_{1\xi} + w_{1z} = 0 \quad (12)$$

with the boundary conditions:

$$p_1 + \eta_0 p_{0z} = \eta_1, \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \eta_{0\tau} + u_0 \eta_{0\xi}, \quad (13)$$

valid at $z = 1$. We also get from the BC at the bottom: $z = 0$, the relation

$$w_1 = -\alpha G_0(\eta_0, \eta_{0\xi}, \dots) \quad (14)$$

where G_0 is the contribution of the leakage function at ϵ^0 order. Using the above result, w_1 can be expressed now as

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi} + \frac{1}{2} \eta_{0\xi\xi\xi})z + \frac{1}{6} z^3 \eta_{0\xi\xi\xi} - \alpha G_0, \quad (15)$$

giving thus all other functions expressed through the fields η_0 and η_1 only, in this order of approximation. Finally eliminating η_1 we obtain the free surface wave equation as

$$2\eta_{0\tau} + \frac{1}{3} \eta_{0\xi\xi\xi} + 3\eta_0 \eta_{0\xi} + \alpha G_0 = 0, \quad (16)$$

with an additional term due to the wave profile dependent bottom leakage function appearing in the well known integrable KdV equation[36], which however spoils the integrability of the system, in general. With a scaling of the variables as $U = 9\eta_0, T = \tau/6$ equation (16) takes a normalized form

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta G_0 = 0, \quad (17)$$

where α is scaled to β and $G_0(U, U_\xi, \dots)$ is an arbitrary smooth function, originating from the wave profile dependent leakage velocity. It is fascinating to note, that the condition, we impose for the fluid velocity at the bottom through a boundary condition with wave profile dependence makes it way to the nonlinear evolution equation at the surface.

Notice that equation (17) is an extension of the KdV equations with arbitrary higher nonlinearity, which in general represents a non integrable system. However an approximate method due to Bogoliubov and Mitropolsky [33, 37, 38] could be applied here for extracting analytic solutions for the wave equation (17), in general, in an implicit form. For explicit analytic solution, one needs to make suitable choices for function G_0 . We focus below on some of such choices with lower order nonlinearities, e.g. $G_0 = U, U^2, U^3, U_\xi^2$ though this set, in principle, can be extended further. We do not put emphasis on the physical meaning for the individual forms of the leakage function, since our main motivation is to compare theoretically the result of the corresponding wave solutions, to identify the case that would induce maximum damping of the wave amplitude. It is intriguing to note, that similar equations for some of the cases considered by us were obtained earlier [31, 33], though in completely different physical set-ups.

In order that this approximation scheme to be consistent with the condition for the validity of (17), it is required that the leakage coefficient β should be a small parameter of order higher than ϵ as $1 \gg \beta \gg \epsilon$.

Introducing a phase coordinate $\phi(\xi, T, \beta) = \sqrt{\frac{N(T, \beta)}{12}}(\xi - \frac{1}{3} \int_0^T N(T, \beta) dT)$, through a time-dependent function $N(T, \beta)$, assumed to vary slowly with time, with two different time scales $t_0 = T, t_1 = \beta T$, we seek a solution of the wave equation following [38]. By expanding $U(\phi, \beta, T)$ in small parameter β as

$$U(\phi, \beta, T) = U_0(\phi, t_0, t_1) + \beta U_1(\phi, t_0) + O(\beta^2), \quad (18)$$

valid for long times, (as large as $T \sim O(1/\beta)$), we obtain using (17) an equation containing different powers of β . Equating coefficients of the same powers of β , equations at different orders are derived, which need to be solved at each order.

B. Case $G_0 = U$

We explore this case with some details for demonstrating the applicability of the Bogoliubov method for solving perturbed KdV equation and for identifying the quantitative trend in the influence of the bottom leakage G_0 on the amplitude of the surface waves. Note that the equation obtained in this case mathematically coincides with the dissipation induced evolution considered in the context of ion-sound waves damped by ion-neutral collisions [33].

Integrating equation (17) for $G_0 = U$, over the whole range of ξ we can solve for the total wave amplitude $I(T) = \int_{-\infty}^{\infty} U d\xi$ and the total intensity of the wave $P(T) = \int_{-\infty}^{\infty} U^2 d\xi$ to get the explicit expressions as $I(T) = I(0) \exp(-\beta T)$ and $P(T) = P(0) \exp(-2\beta T)$, respectively, where $U(\xi, T)$ and its higher order ξ derivatives are assumed to vanish at infinity. It is also evident from the exponentially decaying nature, that the wave intensity is not conserved in time, confirming that the integrability of the perturbed KdV equation (17) in this case is lost due to the leakage we have considered here.

Since estimating the damping of the solitary water waves is the main concern of our problem, we take the following relations as the required initial and boundary conditions: $U(\phi, 0, \beta) = N_0 \text{sech}^2(\phi)$, $U(\pm\infty, T, \beta) = 0$. The lowest order equation takes the form

$$\rho \frac{\partial U_0}{\partial t_0} + \frac{\partial^3 U_0}{\partial \phi^3} - 4 \frac{\partial U_0}{\partial \phi} + \frac{12}{N} U_0 \frac{\partial U_0}{\partial \phi} = 0, \quad (19)$$

where $\rho = \frac{12\sqrt{12}}{N\sqrt{N}}$ with $N(t_1)$ as an arbitrary function of t_1 , except for the initial condition $N(0) = N_0$. Solving this equation we obtain

$$U_0(\phi, t_0, t_1) = N(t_1) \text{sech}^2(\phi), \quad (20)$$

while the β order equation takes the form

$$\frac{\partial U_1}{\partial t_0} + L[U_1] = M[U_0], \quad (21)$$

where,

$$M[U_0] = -\frac{\partial U_0}{\partial t_1} - \frac{\phi}{2N} \frac{\partial U_0}{\partial \phi} \frac{dN}{dt_1} - U_0, \quad L[U_1] = \frac{1}{\rho} \frac{\partial^3 U_1}{\partial \phi^3} - \frac{4}{\rho} \frac{\partial U_1}{\partial \phi} + \frac{12}{N\rho} \frac{\partial(U_0 U_1)}{\partial \phi}. \quad (22)$$

The boundary and initial conditions for U_1 are $U_1(\pm\infty, t_0) = 0, U_1(\phi, 0) = 0$ and it is required that $U_1(\phi, t_0)$ should not behave secularly with t_0 . To eliminate secular behavior of U_1 it is necessary that $M[U_0]$ be orthogonal to all solutions $g(\phi)$ of $L^+[g] = 0$, where the function $g(\phi)$ should satisfy $g(\pm\infty) = 0$. Here L^+ is the operator adjoint to L given by,

$$L^+ = -\frac{1}{\rho} \frac{\partial^3}{\partial \phi^3} + \frac{4}{\rho} \frac{\partial}{\partial \phi} - \frac{12}{\rho} \text{sech}^2(\phi) \frac{\partial}{\partial \phi}. \quad (23)$$

One can show, that the only possible solution of $L^+[g] = 0$, with $g(\pm\infty) = 0$, is in the solitonic form $g(\phi) = \text{sech}^2(\phi)$. Thus from the orthogonality requirement we get

$$\int_{-\infty}^{\infty} \text{sech}^2(\phi) M[U_0] d\phi = 0, \quad (24)$$

which yields a simple first order differential equation for $N(t_1)$, the solution of which is

$$N(t_1) = N(0) \exp\left(-\frac{4t_1}{3}\right), \quad t_1 = \beta T \quad (25)$$

for positive small leakage parameter β at large time T . Therefore we obtain the final result as

$$U = N(t_1) \text{sech}^2 \phi(\xi, t_1) + O(\beta), \quad \phi(\xi, t_1) = \sqrt{\frac{N(t_1)}{12}} \left(\xi + \frac{1}{4\beta} N(t_1) \right). \quad (26)$$

The wave solution of equation (17) thus obtained for $G_0 = U$, shows that the amplitude of the solitary wave would decrease with time following (25).

Recall that similar dissipative soliton solution was derived earlier in many different physical situations [31, 33].

C. Case $G_0 = U^2$

We take up this case for comparison and find that the same Bogoliubov- Mitropolsky method discussed above is applicable also in this case with the wave equation taking the form of a perturbed KdV equation

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U^2 = 0. \quad (27)$$

Notice, that equation (17) with the choice for our leakage velocity function, coincides formally with the dissipation due to friction at the bottom (Chezy law) [31], though for completely different origin.

Using the same approximation technique, details of which we omit, the decay law of the solitary wave amplitude for equation (27) can be derived as

$$N(T) = \frac{N(0)}{\left[1 + \frac{16N(0)\beta}{15} T\right]}. \quad (28)$$

Observe, that in comparison with the linear choice of the leakage velocity the amplitude decay with time becomes weaker in this nonlinear case. To confirm this trend, which is rather anti-intuitive we take up new cases with enhanced nonlinearity and derivatives.

Interestingly, the choice of leakage function as $G_0 = -U_{\xi\xi}$ would lead to very similar decay law (28) and would also coincide formally with the effect of magneto-sonic waves damped by electron collisions [33].

D. Case $G = U^3$

Such a choice of leakage velocity condition with cubic dependence on wave profile would give rise to the equation

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U^3 = 0, \quad (29)$$

representing a new perturbed KdV equation, apparently ignored earlier. The same approximate treatment leads to the decay law of the solitary wave amplitude of (29) as

$$N = \frac{N(0)}{\sqrt{[1 + \frac{32N(0)^2\beta}{35}T]}}, \quad (30)$$

decreasing with time as shown in FIG 2. Since here for cubic nonlinearity we get the decay rate in inverse square root

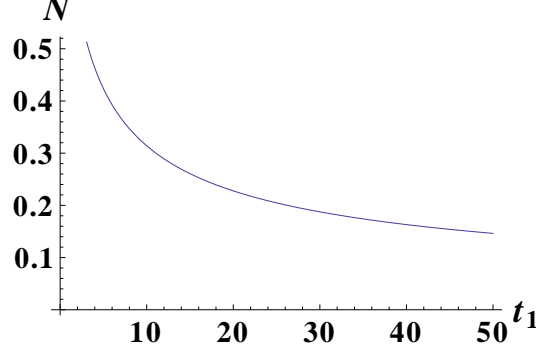


FIG. 2: Plot showing the dependence of the soliton amplitude N on t_1 , for the solution (30) with $N(0) = 1$. The decaying nature of $N(t_1)$ is explicit.

power as seen from (30), we notice again that the same trend of the weaker decay of the soliton amplitude with higher nonlinear dependence of the wave profile on the leakage velocity function, continues confirming the anti intuitive trend noticed above.

E. Case $G = U_\xi^2$

For this choice of the leakage velocity function the perturbed KdV equation reduces to

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U_\xi^2 = 0, \quad (31)$$

apparently not investigated earlier. Through similar procedure we can derive the damped solitary wave amplitude of (31) as

$$N = \frac{N(0)}{\sqrt{[1 + \frac{8N(0)^2\beta}{45}T]}}, \quad (32)$$

which is graphically represented in FIG 3.

Comparing (28) with (32) we may conclude, that the increase of nonlinearity as well as derivatives, of the wave profile in the leakage velocity function weakens the decay rate of the solitonic amplitude. Analyzing the above results for linear and nonlinear choices of G_0 , we may conclude that the leakage with the linear dependence on the profile $G_0 = U$ is the optimal one capable of inducing maximum decay rate on the soliton amplitude as exponential functions, compared to all other cases considered here. Therefore in the next sections we take up this particular case, being the most desirable one, for controlling the surging waves in a decreasing depth scenario.

III. EFFECT OF LEAKAGE BC ON NONLINEAR SHALLOW WATER SURFACE WAVE IN VARIABLE DEPTH BATHYMETRY

Propagation of nonlinear shallow water unidirectional waves over variable depth topography has been studied intensively with rich results [1, 31, 32, 35].

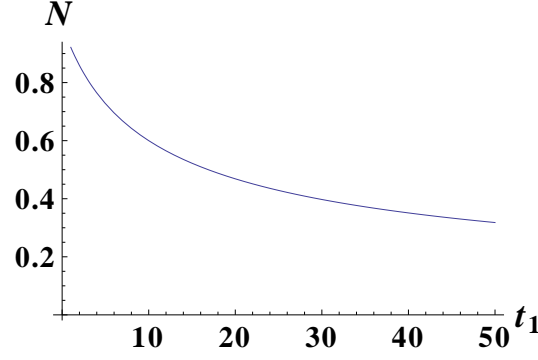


FIG. 3: The solitonic wave amplitude $N(t_1)$ (32) as decays with time t_1 for $N(0) = 1$.

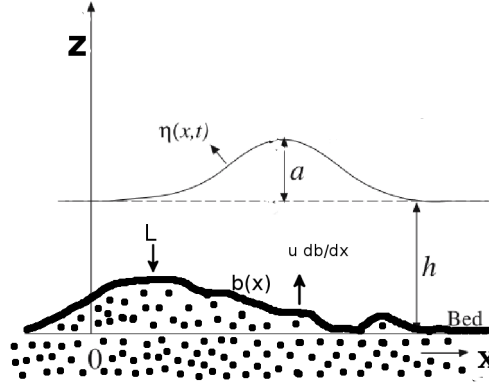


FIG. 4: Solitary wave in a shallow water of slowly varying depth with leakage at the bottom

It is known that the slowly variable depth in comparison to the evolution scale of the wave, can lead to the upsurging wave amplitude, for decreasing depth, which occurs when the wave approaches to the shore. In this section we intend to focus on such a situation due to its potentially hazardous consequences and look for its possible regulation through bottom leakage. Since in the previous section we have identified the maximum damping effect of surface waves for leakage velocity function depending linearly on the wave profile, we will apply this particular leakage condition to achieve maximal damping effect. Therefore we take up the problem of nonlinear wave propagation over shallow water of slowly varying depth in the framework of KdV equation, together with a nontrivial leakage condition at the bottom with a leakage function proportional to the surface wave profile, sensed through a feedback mechanism.

This problem targeted towards controlling the surging waves due to decreasing depth bathymetry, has not received the needed attention.

A. Derivation of nonlinear surface wave evolution equation with slowly variable depth under bottom boundary leakage condition

Under this physical situation one has to start with the same basic dimensionless hydrodynamic equations considered in the previous section as:

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \epsilon[w_t + \epsilon(uw_x + ww_z)] = -p_z, \quad u_x + w_z = 0, \quad (33)$$

together with the surface boundary conditions $p = \eta$, $w = \eta_t + \epsilon u \eta_x$ valid at $z = 1 + \epsilon \eta$. However, the effect of variable depth and the leakage condition enter through a more general boundary condition at the bottom, varying as $z = b(x)$:

$$w = u \frac{db}{dx} - \epsilon g(\epsilon x) G(\eta, \eta_x, \dots). \quad (34)$$

Note that in comparison with the previous case (4) together with the variable depth function an additional leakage function $g(\epsilon x)$ independent of the wave profile η appears with G similar to the feedback leakage function as considered in the previous section. The bathymetry function b is assumed to depend on the small parameter ϵ , such that $b(x) = B(\epsilon x)$. As we have identified in previous section, we assume $G = \eta$ to get the maximum benefit of damping due to leakage. For detailed investigation we introduce a new set of variables

$$\xi = \frac{1}{\epsilon}\chi(X) - t, \quad X = \epsilon x, \quad (35)$$

where $\chi(X)$ will be determined later in equation (38). For solving the above set of equations we would represent the asymptotic solutions as we have used earlier.

We stress again that the hydrodynamic equations involved here are the same as those used in the previous section in dealing with the constant depth problem, except the crucial BC at the bottom.

1. Result at ϵ^0 order

At ϵ^0 order, the above equations are reduced to

$$u_{0\xi} = \chi' p_{0\xi}, \quad p_{0z} = 0, \quad \chi' u_{0\xi} + w_{0z} = 0, \quad (36)$$

together with the boundary conditions $p_0 = \eta_0$, $w_0 = -\eta_{0\xi}$, valid at the surface and $w_0 = 0$, at the variable bottom $z = B(X)$.

Using the above bulk equations and the boundary conditions we obtain

$$p_0 = \eta_0, \quad u_0 = \chi' \eta_0, \quad w_0 = \chi'^2 \eta_{0\xi}(B - z), \quad \chi'^2 = \frac{1}{D(X)}, \quad (37)$$

where $D(X) = 1 - B(X)$ and χ' is the derivative of χ with respect to X . χ can be solved explicitly through the bathymetry function for the right moving wave as

$$\chi(X) = \int_0^X \frac{dX_1}{\sqrt{D(X_1)}}. \quad (38)$$

2. ϵ order approximation

In next order approximation we obtain the set of equations

$$-u_{1\xi} + \chi' u_0 u_{0\xi} + w_0 u_{0z} = -\chi' p_{1\xi} - p_{0X}, \quad p_{1z} = w_{0\xi}, \quad \chi' u_{1\xi} + u_{0X} + w_{1z} = 0 \quad (39)$$

together with the surface boundary conditions

$$p_1 = \eta_1, \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + u_0 \chi' \eta_{0\xi}, \quad (40)$$

and the condition

$$w_1 = u_0 B'(X) - g(X) \eta_0, \quad (41)$$

valid at the variable bottom with $B'(X)$ denoting derivative in X . Our aim is to express other field variables only through the wave functions η_0 and η_1 as

$$p_1 = \eta_1 + \frac{1}{D} \eta_{0\xi\xi} \left[\frac{1}{2} (1 - z^2) + B(z - 1) \right] \quad (42)$$

and

$$\begin{aligned} w_1 = & \left(\frac{B'}{\sqrt{D}} - g \right) \eta_0 + \frac{(B - z)}{\sqrt{D}} \eta_{0X} + (B - z) \left(\frac{\eta_0}{\sqrt{D}} \right)_X + \frac{(B - z)}{D} \eta_{1\xi} + \frac{(B - z)}{D^2} \eta_0 \eta_{0\xi} \\ & - \frac{\eta_{0\xi\xi\xi}}{D^2} \left[B \left(\frac{z^2}{2} - z \right) + \frac{(z - \frac{z^3}{3})}{2} - \frac{B^3}{3} + B^2 - \frac{B}{2} \right]. \end{aligned} \quad (43)$$

Using the above expressions we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + \left(\frac{D'}{2\sqrt{D}} + g\right)\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0. \quad (44)$$

Note that this variable coefficient KdV equation contains explicitly the bathymetry function $D(X)$ linked to the variable depth as well as the function $g(X)$ related to the leakage at the bottom. This variable coefficient KdV equation containing the combined effect of variable depth and the leakage is an important result we have derived here. Different types of variable coefficient KdV like equations were studied earlier for analyzing the possible solutions both in one [39–42] and two dimensions [43, 45].

B. Nature of the solitary wave solution

It is evident that in the absence of the leakage ($g = 0$), our equation (44) would reduce to the KdV equation with variable depth [1, 31, 35]:

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + \left(\frac{D'}{2\sqrt{D}}\right)\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0. \quad (45)$$

When the depth variation occurs in a scale slower than the evolution scale of the wave, the solitary wave solution of equation (45), as is well known, can be expressed as an approximate solution

$$\eta_0 = \frac{A_0}{D} \text{sech}^2\left[\sqrt{\frac{3A_0}{4D^3}}\left(\xi - \frac{D^{-(\frac{5}{2})}A_0X}{2}\right)\right], \quad (46)$$

as given in [35]. Here A_0 is the amplitude of the wave for constant depth ($D = 1$). It is clearly seen that the amplitude of the solitary wave increases as D decreases i.e. the water becomes shallower, showing that such waves would approach the shore with surging amplitude. Note that for exponentially decreasing depth D the growing of wave amplitude will also be exponential. This particular case will be considered in more details in the next section.

It is intriguing to note that for variable bathymetry with uneven depth, irregular depth or periodic topography in place of growing amplitude one gets a damping wave amplitude as explained in [32]. We will be concerned however with the surging waves caused by a smoothly decreasing depth due to their hazardous effects.

Now we will analyze the solution of equation (44) with nontrivial boundary leakage, rewriting it in a more general form

$$a(X)\eta_{0X} + b(X)\eta_0\eta_{0\xi} + c(X)\eta_0 + d(X)\eta_{0\xi\xi\xi} = 0, \quad (47)$$

where we have denoted $a(X) = 2\sqrt{D}$, $b(X) = \frac{3}{D}$, $c(X) = \left(\frac{D'}{2\sqrt{D}} + g\right)$ and $d(X) = \frac{D}{3}$. Dividing (47) by $d(X)$ and defining $\eta_0 = \frac{U}{b_1}$ where $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$ and $c_1 = \frac{c}{d}$, respectively, the equation (47) can be transformed to

$$a_1U_X + UU_\xi + U_{\xi\xi\xi} + \left(c_1 - a_1\frac{b_{1X}}{b_1}\right)U = 0, \quad (48)$$

which in general cannot be solved exactly. However, we may notice, that for a finer balance tuned between the variable depth bathymetry and the controlled leakage velocity function giving the condition

$$g = -\frac{9D'}{2\sqrt{D}}, \quad (49)$$

the last term of (48) vanishes reducing the equation to a more simple form of variable coefficient KdV equation

$$a_1U_X + UU_\xi + U_{\xi\xi\xi} = 0, \quad (50)$$

where $a_1 = \frac{a(X)}{d(X)}$. It is interesting to note, that the tuning condition (49) relating the leakage function with the bathymetry function is exactly same as the solvability condition used in [40] for obtaining analytic solutions of a general variable coefficient KdV equation, considered in a formal mathematical setting.

Defining a new coordinate $T = \int \frac{\sqrt{D(X)}}{6} dX$ equation (50) can be transformed into the standard constant coefficient KdV equation

$$U_T + UU_\xi + U_{\xi\xi\xi} = 0, \quad (51)$$

admitting the well known solitary wave solution $U = N_0 \text{sech}^2[\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3} \int \frac{\sqrt{D}}{6} dX)]$. Expressing in terms of the original field variable we get finally the wave solution

$$\eta_0 = \frac{D^2}{9} N_0 \text{sech}^2[\sqrt{\frac{N_0}{12}}(\xi - V(X)), \quad V(X) = \frac{N_0}{3} \int \frac{\sqrt{D}}{6} dX \quad (52)$$

with the depth function $D(X)$ and leakage velocity function $g(X)$ are tuned as (49). Note that for decreasing depth D , which without leakage would make the wave amplitude to surge as in (46), due to the controlled tuning of the leakage the resultant solitonic wave function would suffer a damping of its amplitude as evident from (52). Moreover the solitonic wave flattens down with a change in its velocity along its propagation (see FIG 5). Thus we have achieved control over a surging wave approaching to the shore by inducing combination of feedback and a controlled tuning of the the leakage at the bottom.

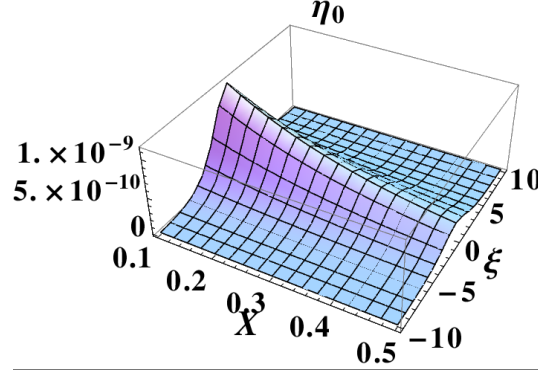


FIG. 5: 3D plot of the solitary wave solution (52) in the (ξ, X) plane. For demonstrating the nature of the solution, we have assumed $N(0) = 1$ $\alpha = 0.1$, $g = \exp[-X]$, showing exponential damping of the wave amplitude with a change in its width and velocity along its propagation.

IV. APPLICATION OF THE EXACT RESULT TO REAL NEAR SHORE BATHYMETRY

In the previous sections, we have first discussed the effect of wave profile dependent leakage to the solitary wave amplitude at constant water depth. Applying similar mathematical procedure to a slowly changing bathymetry, we have derived next a variable coefficient KdV equation containing terms due to both leakage and variable depth. Though in general such equations are non integrable, a finer balance between the leakage and variable depth function miraculously solves the equation exactly, giving a solitary wave like solution. Its amplitude, which without leakage would increase giving surging effects, decreases as the wave moves towards the shallower region. These theoretical findings of exact nature with an intension to control near shore surging waves, by creating artificial leakage, would gain ground when it is implemented to a real sea shore bathymetry. Therefore, in this section we apply previously obtained exact results to a near shore bathymetry in order to see the effectiveness of our findings.

One should remember the fact that according to the estimates of the United Nations in 1992, more than half of the population lives within 60 km of the shoreline. Urbanization and rapid growth of coastal cities have also been dominant population trends over the last few decades, leading to the development of numerous mega cities in all coastal regions around the world.

Our study region is the coastal zone of Chennai district of the Tamil Nadu state, in southeast coast of India which was one of the worst affected areas during 2004 Indian Ocean tsunami. A Coastal Vulnerability Index was developed for this region [34] using eight relative risk variables including near shore bathymetry to know the high and low vulnerable areas. According to one of those risk variables, bathymetry at about 29.11 km of coastline in that area has a high risk rating having high vulnerability, while about 18.55 km of coastline has medium risk rating and about 10.54 km shows low risk rating, which are displayed in FIG 6.

The depth contour of Chennai coastline, which is constructed from the Naval Hydrographic Charts for 2002, is also given in [34] and is displayed in FIG 7. Now to implement our exact results on this coastline, we chose one of the high risk points (N 13° 10.5' - E 80° 18.75') , which is denoted by the red line in FIG 6.

We have drawn the near shore bathymetry following the depth contour (FIG.7) of this shoreline point along the latitude which is given as FIG.8. This diagram shows that at the near shore region, the depth function flattens down

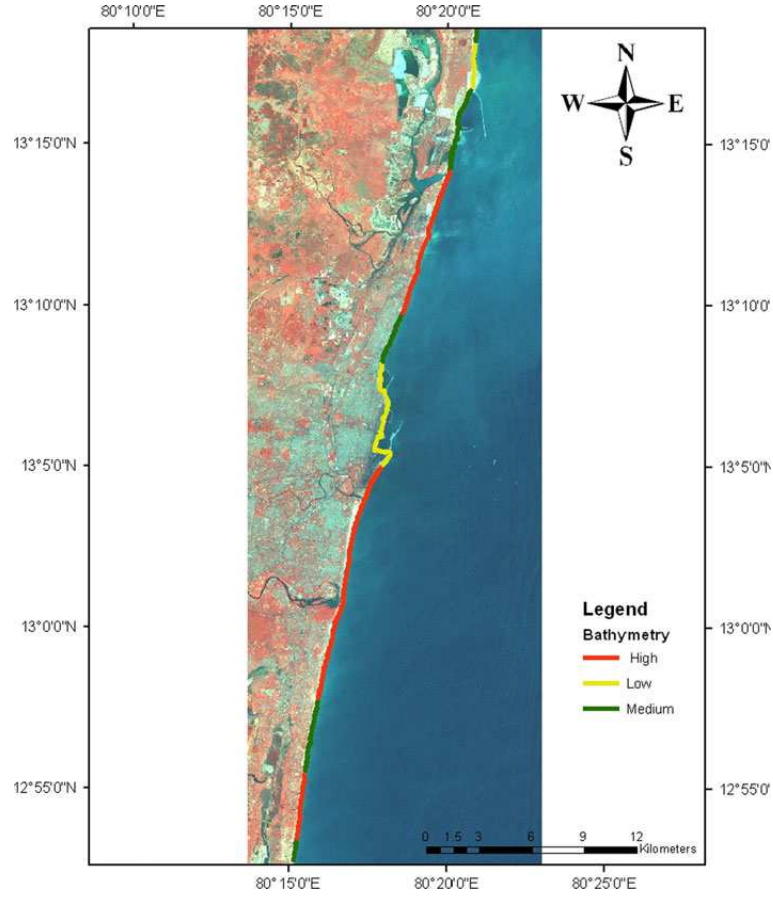


FIG. 6: Risk zones of Chennai coastline bathymetry ,taken from [34] with permission

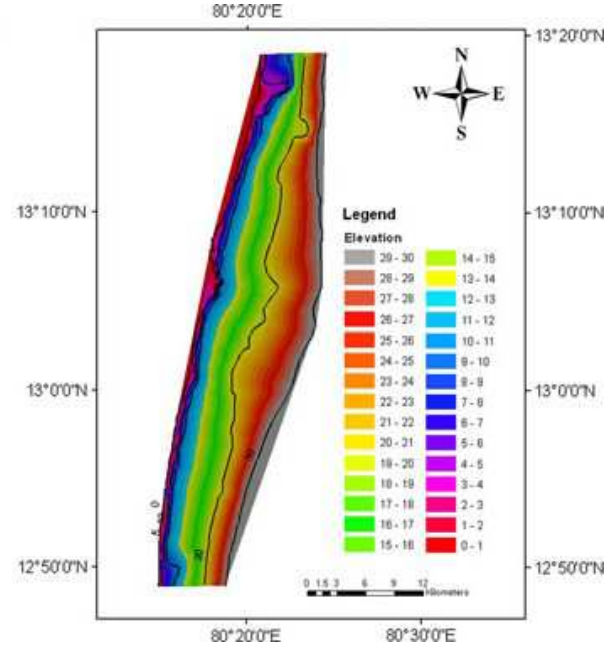


FIG. 7: Depth contour of the Chennai coastline,taken from [34] with permission

denoting a slow variation along X . Hence the soliton gets enough time to evolve to give the surging effects. Note

also that, the variation along X is in km whereas variation along $D(X)$ is in meter. Hence the depth function is very slowly varying which is consistent with our theoretical assumptions.

Note that in the absence of the leakage at the bottom, the solitary wave amplitude would increase following (46) with the amplitude as

$$A_1 = \frac{A_0}{D}. \quad (53)$$

We see from FIG.8 that as the wave approaches the near shore region, the depth function flattens out and therefore the soliton amplitude A_1 develops rapidly to give surging effects. Now if at a certain position in the near shore bathymetry, an artificial leakage following our theoretical findings (52), is turned on then the amplitude would decrease as $A_2 = \frac{N_0 D^2}{9}$, where N_0 is a free constant. The effectiveness of the amplitude decay of the solitary waves by the leakage would be stronger, if the leakage starts at a longer distance away from the shore. Note that the amplitude starts growing rapidly at 1.2 km away from the shore, from where the depth function starts flattening.

Therefore, if a solitary wave of amplitude of nearly 1 meter starts approaching towards the shore from around 10.5 km, then it would ultimately grow to a surging wave of amplitude ~ 30 meter at the coast. It is obvious that such a huge wave will produce devastating effects on coastal habitation and costly installations.

However if we implement now an artificial leakage based feedback method linked to the surface wave profile as discussed in the previous section with exact result (52) the surging amplitude would decrease when propagating towards the shore. with damping amplitude given as (52)

$$A_2 = \frac{N_0 D^2}{9}, \quad (54)$$

where N_0 is a free constant which is chosen following the actual physical condition as described earlier. One checks that if the leakage is implemented in a region of 0.9 km from the shore (at the point Q_1 in FIG.9), the wave amplitude of 1 meter which would otherwise increase to 30 meter without any leakage (denoted by point D in FIG.9), would decrease to an amplitude of ~ 1.23 meter (denoted by point A in FIG.9), where we have chosen $N_0 = 11.60$.

If the leakage installation is implemented from a nearer point from the shore, the wave amplitude decrease would also be less which is also displayed in FIG.9. For optimal estimation however, the cost effectiveness and the concrete requirements should be taken into account in deciding the range of such proposed installations. The main emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea coast against the danger of tsunami like waves. The options known for the protection of the Chennai coast area are dune afforestation, mangrove restoration and management, periodic beach nourishment and building seawalls and groins etc. Our control mechanism for the possible management of the potentially hazardous near shore waves, proposed here, could be a new option, which may be implemented only in limited strategic areas surrounding costly installations, for reducing the intensity of the approaching wave to a safer limit.

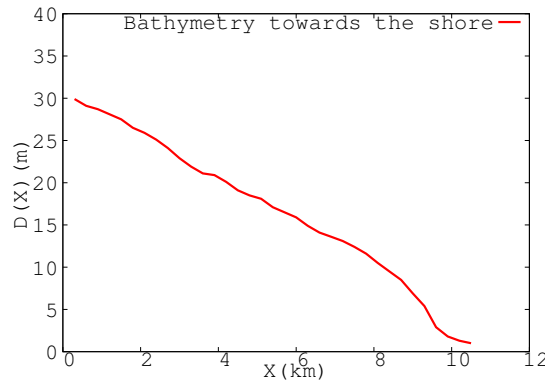


FIG. 8: Bathymetry towards the shore of the shoreline point (N 13° 10.5' - E 80° 18.75')

The same methodology can be applied to another high risk point (N 13° 0' - E 80° 16.2') at the shoreline, the near shore bathymetry of which is shown in FIG 10. The increase of amplitude without leakage, and its damping due to leakage is explicit in FIG.11.

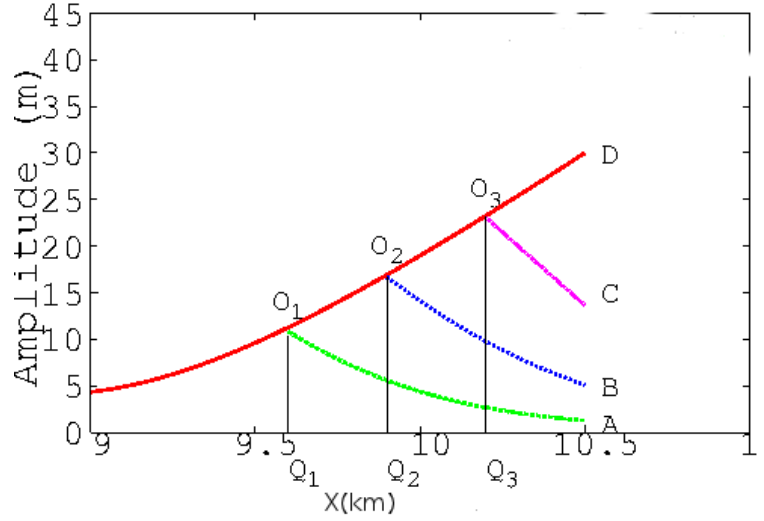


FIG. 9: Surging amplitude A_1 without leakage moving towards the shoreline point (N $13^\circ 10.5'$ - E $80^\circ 18.75'$) and growing upto the point D (30 m) following eq.(53). Figure also demonstrates the damping of the amplitude A_2 due to leakage following eq. (54) . Installations of the leakage starting from different points to the shore Q_1 (9.6 km), Q_2 (9.9 km), and Q_3 (10.2 km), would damp the amplitude A_2 to different values (A (1.23 m), B (5.15m) and C (13.5m) respectively). N_0 , a free constant appearing in eq. (54) is chosen as 11.60, 46.29 and 122.89 respectively at these points. It is evident that, the further the leakage is from the shore, the more the decay of the amplitude.

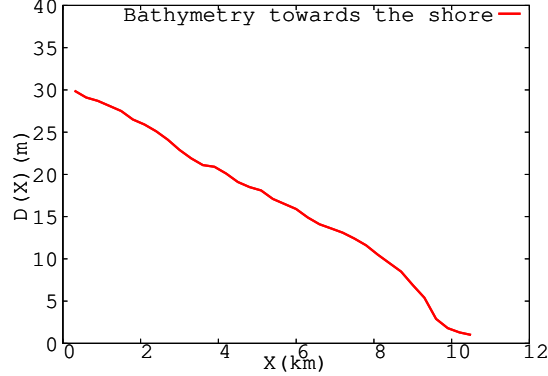


FIG. 10: Bathymetry towards the shore of the shoreline point (N $13^\circ 0'$ - E $80^\circ 16.2'$)

V. CONCLUDING REMARKS

The focus of our investigation is an innovative possibility of controlling the intensity of near shore surging waves including tsunamis and bore waves, by inducing damping effect through a specially designed leakage mechanism at the water bed.

The majority of the earlier studies, concentrated on the damping of the waves occurring due to natural effects like viscosity, bottom roughness, sand porosity etc. In contrast, our main motivation here is to analyze the impact of artificially created bottom boundary condition on the swelling wave approaching the shore, with an aim to reduce the hazardous effect of such near shore wave phenomenon.

Our crucial observation is, that the surging of approaching waves caused by decreasing water depth bathymetry may be thought of to be triggered by effective vertical fluid flow proportional to the gradient of the depth profile, acting as a virtual *source* emerging from the bottom. Our key idea for controlling the growing amplitude of the surface wave is to counter this source by an effective *sink* through such leakage mechanism creating a downward fluid velocity.

We have considered the propagation of an unidirectional, shallow water, nonlinear free surface gravity wave based on the basic hydrodynamic equations at the shallow water regime and identified first, that a feedback leakage function at the bottom, dependent linearly on the surface wave profile, could induce maximum desirable damping effect on the amplitude of the surface wave. This knowledge is then applied to the problem of regulating the surging solitary

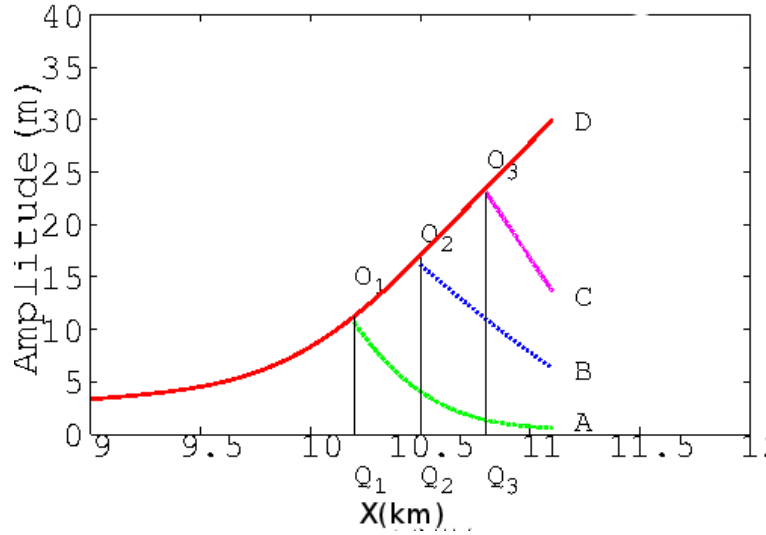


FIG. 11: Surging amplitude A_1 without leakage moving towards the shoreline point (N $13^\circ 10.5'$ - E $80^\circ 18.75'$) and growing upto the point D (30 m) following eq.(53). Figure also demonstrates the damping of the amplitude A_2 due to leakage following eq. (54) . Installations of the leakage starting from different points Q_1 (10.2 km), Q_2 (10.5 km) and Q_3 (10.8 km), would damp the amplitude A_2 to different values (A (0.53m), B(6.24m) and C(13.57m) respectively). N_0 , a free constant appearing in eq. (54) is chosen as 5.44, 56.91 and 122.89 respectively at these points. It is evident that, the further the leakage is from the shore, the more the decay of the amplitude.

waves propagating towards the shore, due to the slowly decreasing depth. The corresponding evolution equation for the combined effect of leakage and the variable bathymetry turns out to be in the form of a variable depth KdV equation, different from the variable coefficient KdV equation obtained earlier. Though in general this is a non-integrable system, we have found, that for a controlled tuning between the topography and leakage velocity function, the equation becomes exactly solvable, allowing solitary wave solutions with damping amplitude.

A strong point of our result is its exact nature, which allows one to access precise and finer effects and make more accurate predictions. We have applied the result obtained to real data from the bathymetry map of the tsunami prone near shore regions on the Bay of Bengal in India and tested the implications, range and predictions of our theoretical result. As shown by the real bathymetry, the more extensive installations starting from a further distance into the sea would result to a more effective control of the incoming surging waves. However, the cost effectiveness and the concrete requirements should be taken into account in deciding the range of such proposed installations. The main emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea cost against the danger of tsunami like waves. Therefore the control mechanism for the possible management of the potentially hazardous near shore waves, proposed here, may be implemented only in limited strategic areas surrounding costly installations, for reducing the intensity of the approaching wave to a safer limit.

We have studied also various possible extensions of the leakage boundary conditions and their corresponding effects in modifying the nature of the surging solitary waves which might be of practical importance in different other situations.(This material is included as the appendix.)

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- [1] A. Kundu (Ed.) *Tsunami and Nonlinear Waves* (Springer, Berlin, 2006).
 - [2] E. N Dolgoplova *Water Resources***40** (2013)16
 - [3] H. C. Miah *J.Hydraulic. Res***43**(2005) 234
 - [4] S. Furuyama & H Chanson *Coastal Eng. J* **52** 215
 - [5] A Rudloff et. al *Nat.Haz.Earth.Sys. Sci* **9**(2009) 1381
 - [6] S. Nayak & T.S Kumar *Int. Archives.Photogrammetry* **XXXVII**(2008)B4
 - [7] J Lauterjung, U Munch, A. Rudloff *Nat.Haz.Earth.Sys. Sci***10**(2010)641
 - [8] D.E Barrick *Remote Sensing of Environment* **8**(1979) 353
 - [9] G Blewitt et.al *J. Geod* **83** (2009) 335
 - [10] E. Marris *Nature* **433** (2005) 3
 - [11] L.T Dauer et.al *J. Nuclear Medicine***52** (2011)9

- [12] T Takemura et.al *SOLA* **7**(2011)101
- [13] K. Kathiresan & N. Rajendran *Estuarine, Coastal and Shelf Sciences* **65** (2005) 601
- [14] N.K Liang, J.S Huang & C.F Li *Ocean Engineering***31**(2004) 43
- [15] W.Koo *Ocean Engineering* **36** (2009) 723
- [16] L. Martinelli, P. Ruol & B. Zanuttigh *Applied Ocean Research* **30** (2008)199
- [17] C. Michailides & D. C Angelides *Applied Ocean Research* **35** (2012) 77
- [18] G. Taylor *Proc. R. Soc. A* **231** (1955) 466; G.I. Taylor, *Admiralty Scientific research Dept. ATR/Misc/1259* 1943.
- [19] D. L. Hammond *United States Patent* (1974) 3,785,159
- [20] A. M Fridman et.al *Physics-Uspexhi* **53**(8) (2010) 809
- [21] J.D. Murray, *J. Geophys. Res.* **70**, 10 (1965). (and references therein).
- [22] R.O. Reid and K. Kajiura, *Trans. Am. Geophys. Union* **38**, 662 (1957).
- [23] J.A. Putnam, *Trans. Am. Geophys. Union* **30**, 349 (1949).
- [24] J.M. Hunt, *J. Geophys. Res.* **64**, 437 (1959).
- [25] T. Yamamoto, H.L. Koning, H. Sellmeijer and E.V. Hijum, *J. Fluid. Mech.* **87**, 193 (1977).
- [26] P.L. Liu and R.A. Dalrymple, *Coastal Engineering* **8**, 33 (1984).
- [27] C.C. Mei, M. Stiassnie and K.P. Yue Dick, *Theory and Applications of Ocean Surface Waves: Nonlinear aspects* (World Scientific, 2005).
- [28] P.L. Liu and I.C. Chan, *J. Fluid. Mech.* **579**, 467 (2007).
- [29] P.L. Liu, G. Simarro, J. Vandever and A. Orfila, *Coastal Engineering* **53**,181 (2006).
- [30] P.L. Liu and J. Wen, *J. Fluid. Mech.* **347**, 119 (1997).
- [31] L.A. Ostrovsky, *Int. J. Nonlin. Mechanics* **11**, 401 (1976).
- [32] O. Nakoulima, N. Zahibo, E. Pelinovsky, T. Talipova, and A. Kurkin, *Chaos* **15**, 037107 (2005).
- [33] E. Ott and R.N. Sudan, *Phys. Fluids* **13**, 6 (1970).
- [34] A.A Kumar and P.D Kunte *Nat Hazards* (2012) **64**, 853 (2012).
- [35] R. S Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, (Cambridge University Press, Cambridge, 1997) and references therein.
- [36] D.J. Korteweg and G. de Vries, *Philos. Mag.*, **39**, 422(1895).
- [37] E. Ott and R.N. Sudan, *Phys. Fluids* **12**, 11 (1969).
- [38] N.N. Bogoliubov and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach Science Publishers, Inc, New York, 1961).
- [39] Zhi-Yuan Sun, Yi-Tian Gao, Ying Liu and Xin Yu, *Phys. Rev. E* **84**, 026606 (2011).
- [40] Xin Yu, Yi-Tian Gao, Zhi-Yuan Sun and Ying Liu, *Phys. Rev. E* **83**, 056601 (2011).
- [41] D. Kaya and M. Aassila, *Phys. Lett. A* **299**, 201 (2002)
- [42] X. Zhao, D. Tang and L. Wang, *Phys. Lett. A* **346**, 288 (2005)
- [43] Y.Z Peng, *Phys. Lett. A* **351**, 41 (2006)
- [44] A. Mukherjee and M.S. Janaki, *Phys. Rev. E* **89**, 062903 (2014).
- [45] M. J. Ablowitz and D. E. Baldwin, *Phys. Rev.* **86**, 036305 (2012).
- [46] A.G. Johnpillai, C.M. Khalique and A. Biswas, *Appl. Math. Comp.* **216**, 3114 (2010).
- [47] V.P Ruban, *Phys. Lett. A* **340**, 194 (2005)

VI. APPENDIX: EXTENSION OF BOUNDARY LEAKAGE CONDITION WITH VARIABLE BATHYMETRY

Though we have achieved our major goals in taming the surging waves as reported in the main text, we consider below few extensions of this result for understanding the effect of bottom boundary leakage condition on the surface wave solution, which might be of applicable interest in other physical situation. In particular ,we have investigated

- A) Leakage function at the bottom with a combination of both wave profile dependent and independent functions,
- B) Leakage condition linked to effective zero fluid velocity at the bottom with the specific bathymetry profile.
- C) Leakage function related to time.

All the studies yielding analytic result of different nature though all of them having the effect of amplitude damping of the waves, surging otherwise due to decreasing depth bathymetry.

A. Leakage function at the bottom with a combination of both wave profile dependent and independent functions

In our previous paper [44], we considered the leakage function to be independent of the wave profile that yielded a forced KdV like equation as the surface wave equation. Its solitary wave solution exhibits phase modification leading its velocity to change whereas the amplitude remaining constant. In order to explore the effect of the bottom leakage

on the solitary wave amplitude we have considered in the main text, the leakage function to be dependent on the free surface wave profile which exhibited damping of amplitude.

Now in this section, we have extended the problem such that the leakage velocity at the bottom depends both on the wave profile dependent and independent functions as

$$w = u \frac{db}{dx} - \epsilon g(\epsilon x) G(\eta, \eta_x, \dots) + \epsilon C(X). \quad (55)$$

on $z = B$. Here the second term in (55) is the wave profile dependent term whereas the third one is the wave profile independent term.

As we have mentioned we assume $G = \eta$ to get the maximum benefit of damping due to leakage. After a bit of mathematical calculations we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + \left(\frac{D'}{2\sqrt{D}} + g\right)\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = -C(X). \quad (56)$$

Note that this variable coefficient KdV equation contains explicitly the bathymetry function $D(X)$ linked to the variable depth as well as the function $g(X)$ and $C(X)$ related to the leakage at the bottom.

Now after applying the same balancing condition (49) the equation can be transformed into

$$a_1 U_X + U U_\xi + U_{\xi\xi\xi} = -E_1 \quad (57)$$

where we have denoted $a_1(X) = \frac{6}{\sqrt{D}}$, $E_1 = \frac{27C(X)}{D^3}$ and $\eta_0 = \frac{U}{b_1}$.

Defining a new coordinate $T = \int \frac{\sqrt{D(X)}}{6} dX$ equation (57) can be transformed into the standard constant coefficient KdV equation with a forcing term

$$U_T + U U_\xi + U_{\xi\xi\xi} = E_1(T), \quad (58)$$

admitting the well known solitary wave solution $U = N_0 \text{sech}^2[\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3} \int \frac{\sqrt{D}}{6} dX) - f(T)] - \int E_1 dT$.

Expressing in terms of the original field variable we get finally the wave solution

$$\eta_0 = (D^2/9)[N_0 \text{sech}^2\{\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3} \int \frac{\sqrt{D}}{6} dX - f(T))\} - \int E_1 dT] \quad (59)$$

where $\frac{\partial^2 f(T)}{\partial T^2} = -E_1$.

Note that if we neglect the wave profile independent part $C(X)$, then automatically we get $C_1 = E_1 = F = 0$ and $f = \text{constant}$. Thus the solution (59) converges to the solution of earlier case (52).

B. Balancing through effective hard bottom condition with leakage giving exact result

Here we stick to a particular choice of decreasing bathymetry $D = \exp(-\sigma X)$, for the wave approaching to the shore. Such solitary waves without any leakage condition would result to an exponentially surging waves carrying potential hazards. Our aim here would be to control such wave through bottom leakage condition inducing necessary damping. For this purpose we consider a different balancing effect of the leakage condition, obtained from an effective hard bottom condition amounting to the vertical fluid velocity at the water bed w to be zero. This leads at the leading order to $w_0 = 0$, $w_1 = u_0 B'(X) - g(X)\eta_0 = 0$, at $z = B$, which gives a new balance between the leakage and the variable depth function as $g = -\frac{D'}{\sqrt{D}}$ at $z = B$. For this effective hard bottom condition, we follow again similar mathematical procedure as presented in the previous section, which leads to the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} - \frac{D'}{2\sqrt{D}}\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0, \quad D = D(X). \quad (60)$$

Note that this variable coefficient KdV equation is different from the variable bathymetry equation (45) obtained earlier [35]. As such this equation is also difficult to solve analytically. However interestingly for a special choice of bathymetry function $D = \exp(-\sigma X)$, with D decreasing with the increase of X , which is consistent with the wave propagating towards shallower region, we can find an exact wave solution for equation (60).

Dividing equation (60) by $\frac{D}{3}$ and redefining the field as $\eta_0 = \frac{D^2}{9}H$, the equation with our specific choice of D can be converted to

$$6 \exp(\sigma X/2) H_X + H H_\xi + H_{\xi\xi\xi} = \left(\frac{21\sigma}{2}\right) \exp(\sigma X/2) H. \quad (61)$$

Defining a new coordinate variable as $T = -\frac{\exp(-\sigma X/2)}{3\sigma}$, equation (61) can be transformed now to a convenient form of the so called concentric KdV equation

$$H_T + H H_\xi + H_{\xi\xi\xi} + \frac{7}{2T} H = 0, \quad (62)$$

which is a known integrable equation derivable from the hydrodynamic equations with cylindrical symmetry [35]. An exact solution of the variable coefficient KdV equation (62) is presented in [46] in the rational form as $H = \frac{(c - \frac{5}{2}\xi)}{T}$.

Using the relation with our original field: $\eta_0 = \frac{D^2}{9}H$ and reverting to our old coordinates ξ, X we can transform back the solution to obtain the required exact solution for the surface wave

$$\eta_0 = -\frac{\sigma}{3} \left(c - \frac{5\xi}{2}\right) \exp(-3\sigma X/2), \quad (63)$$

with an arbitrary constant c . Note that this is a rational solution, not of solitonic type and it behaves differently for different values of ξ . For $\xi < \frac{2c}{5}$, $\eta_0 < 0$, for $\xi > \frac{2c}{5}$, $\eta_0 > 0$, while at $\xi = \frac{2c}{5}$, $\eta_0 = 0$ (see Figure 6). Solution (63) shows that, the amplitude decays down due to the exponential damping factor, as the wave propagates along the positive X direction. Thus the surging waves are controlled to damping wave through balancing with the leakage at the bottom as we have aimed at. At $\xi \rightarrow \pm\infty$ the wave profile shows divergent nature. However since our intention is to consider the wave propagation towards the shore the damping effect obtained along X is the relevant factor.

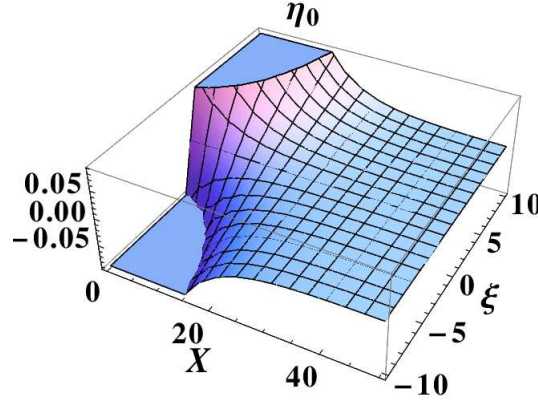


FIG. 12: 3D plot of the exact wave solution (63) in the ξ, X plane with exponentially decreasing depth with X and a bottom leakage with $\sigma = 0.1$. The amplitude decay with the distance traveled along X , is evident. The divergent nature of the solution in ξ can be detected from the figure.

C. Time dependent leakage:

In all the previously discussed cases, the leakage function g is assumed to depend slowly on the space variable x as $g(\epsilon x)$. As an extension to the problem, we consider here a special kind of leakage function, which depends slowly on time, as $g(\epsilon t)$. When the bottom boundary depends on time, then the two dimensional potential flows of an ideal fluid with a free surface are considered in [47].

Here, the bottom boundary condition at the variable bathymetry $z = B(X)$ becomes

$$w = u \frac{db}{dx} - \epsilon g(\epsilon t) G(\eta). \quad (64)$$

where the leakage function g depends slowly on time t . As we have mentioned we assume $G = \eta$ to get the maximum benefit of damping due to leakage. For detailed investigation we introduce a new set of variables

$$\xi = \frac{1}{\epsilon} \chi(X) - t, \quad X = \epsilon x, \quad \Theta = \epsilon t \quad (65)$$

Note that, here we have introduced a new slow time variable Θ which also depends slowly on time. After a bit of calculations we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + 2\eta_{0\Theta} + \left(\frac{D'}{2\sqrt{D}} + g(\Theta)\right)\eta_0 + D\eta_{0\xi\xi\xi} = 0. \quad (66)$$

Note that two extra terms arise due to the slow time Θ which can be canceled in the following way.

Let us consider a new transformation $\eta_0 = f(\Theta)\phi(\xi, X)$. We consider $g(\Theta)$ to be such that the extra two terms which arose due to the slow time Θ cancels each other such that

$$2\eta_{0\Theta} + g(\Theta)\eta_0 = 0 \quad (67)$$

which finally gives $f = A \exp^{-\frac{1}{2} \int g d\Theta}$, where A is a constant. The equation satisfied by the function ϕ is nothing but that obtained by Johnson (45). Hence using their solution (46) as given in [35] the final solution can be written as

$$\eta_0 = \frac{A}{D} \exp^{-\frac{1}{2} \int g d\Theta} \operatorname{sech}^2 \left[\sqrt{\frac{3A_0}{4D^3}} \left(\xi - \frac{D^{-(\frac{5}{2})} A_0 X}{2} \right) \right], \quad (68)$$

The dynamics of the solution (68) can be explained like follows. As the wave propagates towards the shallower region, due to the factor $1/D$ the wave amplitude increases, whereas due to the exponentially decaying factor, which depends on time the amplitude increase is compensated to some extent. But the leakage function $g(\epsilon t)$ should be synchronized in such a way that as the wave starts increasing it starts working. Such physical mechanism and installations can be used in the other physical situations as required.